Critical behavior of the *XY* model on growing scale-free networks

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Applying the histogram reweighting method, we investigate the critical behavior of the *XY* model on growing scale-free networks with various degree exponents λ. For $\lambda \leq 3$, the critical temperature diverges as it does for the Ising model on scale-free networks. For $\lambda = 8$, on the other hand, we observe a second-order phase transition at finite temperature. We obtain the critical temperature $T_c = 3.08(2)$ and the critical exponents $\tilde{\nu} = 2.62(3)$, $\gamma/\tilde{\nu} = 0.127(4)$, and $\beta/\tilde{\nu} = 0.442(2)$ from a finite-size scaling analysis.

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I. INTRODUCTION

The various spin models such as the Ising model, the *XY* model, and the Potts model have been studied on regular lattices [1,2]. The theoretical and numerical studies of these models on complex networks [3–6] also have demonstrated various universal critical behaviors which are determined by the structure of a network and the symmetry underlying a model [7–22].

The study of scale-free networks has been a popular topic recently because many real networks fall into this category: social [23–25], Internet [26–29], biological [30–34], financial [35–38], and geological [39] networks. The scale-free network exhibits a power-law degree distribution $P(k) \sim k^{-\lambda}$, where k is the degree (number of links) of a node and λ is called the degree exponent. The power-law behavior of scale-free networks originates from the growth of networks and preferential attachment [40]. Preferential attachment means that a new node links to a node with uneven probability proportional to the degree of an existing node. As a result, highly linked nodes (hubs) are made by preferential attachment on the growth of networks. The first realization of the scale-free network is the Barabási-Albert (BA) network [40]. Even though the BA network has been successful in explaining many real networks, the degree exponent is fixed at $\lambda = 3$. The growing scale-free network, which is a simple modification of the BA network, has been introduced where the degree exponents can be varied in the domain $\lambda > 2$ [41].

On a regular lattice, both the Ising and *XY* models exhibit mean-field behavior above a upper critical dimension $d_u = 4$. The Ising and *XY* models on small-world networks have been studied analytically and numerically [7–10,19,20]. The critical behavior is characterized by the standard mean-field behavior, whereas the Ising model on the scale-free networks exhibits nontrivial critical behavior, depending on the value of λ [11–18]: For $2 < \lambda \leq 3$, only an ordered phase exists at finite temperatures. For $3 < \lambda \leq 5$, an order-disorder phase transition exists with nontrivial critical exponents. The standard mean-field behavior then emerges for $\lambda > 5$. However, as far as we know, the *XY* model on scale-free networks has not been investigated yet. We have also observed the standard mean-field behavior for $\lambda > 5$ from a preliminary simulation of the *XY* model on static scale-free networks [42]. It is thus interesting to find whether the *XY* model on growing scale-free networks exhibits mean-field behavior or not if $\lambda > 5$.

In this paper, we perform Monte Carlo simulations for the *XY* model on growing scale-free networks to establish the

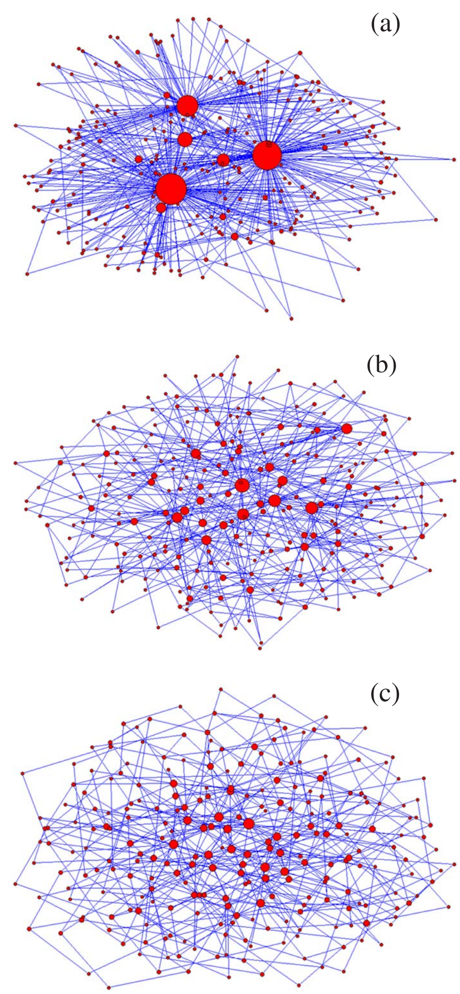

FIG. 1. (Color online) The typical growing scale-free networks with 256 nodes and $q = 2$, where the size of circle is proportional to the degree of k: (a) $\lambda = 2.1$, (b) $\lambda = 3.0$, and (c) $\lambda = 8.0$.

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critical behavior by measuring the magnetization, the susceptibility, and the Binder cumulant. For the high-precision finite-size scaling analysis in Monte Carlo simulations, huge computing resources are needed, because many accurate simulations in the vicinity of the critical temperature should be performed. Using the histogram reweighting method [43–48], we can efficiently manage the numerical data at many temperatures in the vicinity of the critical temperature. We find that no phase transition at finite temperature exists for $\lambda \leqslant 3$. However, our results for $\lambda = 8$ indicate that the *XY* model on growing scale-free networks does not exhibit the standard mean-field behavior.

II. BACKGROUND

A. Model

We study the classical *XY* model whose Hamiltonian is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j), \tag{1}$$

where the sum runs over all nearest-neighbor pairs with the ferromagnetic coupling constant $J = +1$ and ϕ_i is the angular direction on the *XY* plane ranging from $-\pi$ to π.

In the growing scale-free network introduced by Dorogovtsev and co-workers [41], the number of nodes continues to grow by preferential attachment of a newly added node to the existing nodes, until the total number of nodes is equal to the desired system size N. For the fixed value of the degree exponent λ and the average degree of nodes $2q(=\langle k \rangle)$, we have an initial core network where $(q+1)$ nodes are completely connected. Then, a newly added node makes q links to a randomly selected different existing node i with connection probability Π_i defined as

$$\Pi_i = \frac{(k_i + qa)}{\sum_j (k_j + qa)}, \tag{2}$$

where $a(>-1)$ is a variable related to the degree exponent $a = \lambda - 3$ and the sum runs over all existing nodes. In this way we can construct the growing scale-free network following a power-law degree distribution $P(k) \sim k^{-\lambda}$ with $\lambda > 2$. Figure 1 shows the typical growing scale-free networks with 256 nodes and $q = 2$ for $\lambda = 2.1$, 3.0, and 8.0, respectively.

B. Finite-size scaling

The magnetization m, the susceptibility χ, and the Binder cumulant U [49,50] are defined as follows:

$$m = \frac{1}{N} \left| \sum_i e^{j\phi_i} \right|, \tag{3}$$

$$\chi = \frac{N}{T}(\langle m^2 \rangle - \langle m \rangle^2), \tag{4}$$

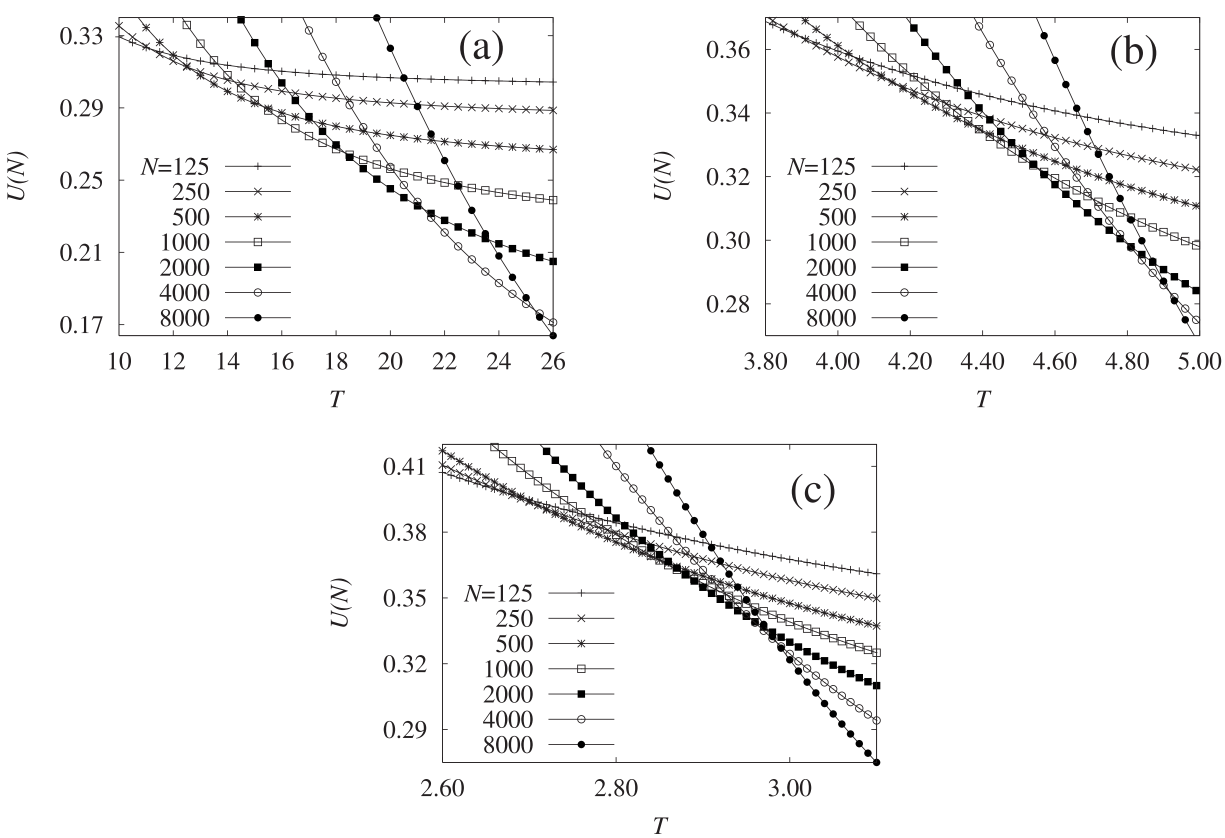

FIG. 2. Plots of the Binder cumulant $U(N)$ as a function of temperature T: (a) $\lambda = 2.1$, (b) $\lambda = 3.0$, and (c) $\lambda = 8.0$.

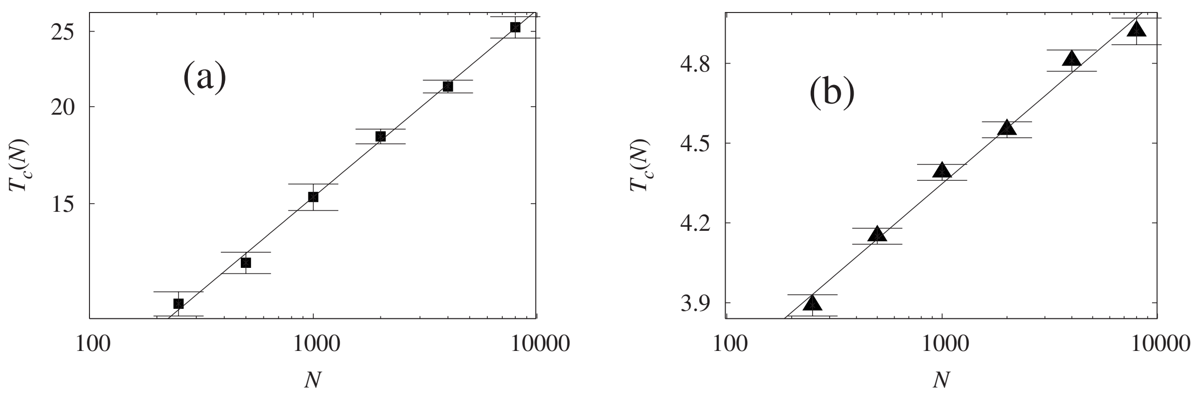

FIG. 3. Plots of the pseudocritical temperature $T_c(N)$ as a function of N: (a) log-log plot for $\lambda=2.1$ and (b) semilogarithmic plot for $\lambda=3.0$.

$$U = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}, \tag{5}$$

where $j=\sqrt{-1}$. For the finite-size scaling analysis, we use the correlation volume exponent $\tilde{\nu}$ instead of the correlation length exponent ν because the linear system size L is not defined in networks [19]. Then, the finite-size scaling forms of the magnetization, the susceptibility, and the Binder cumulant for the network having N nodes are given by

$$m(N) = N^{-\beta/\tilde{\nu}}\tilde{m}(N^{1/\tilde{\nu}}t) \quad (t<0), \tag{6}$$

$$\chi(N) = N^{\gamma/\tilde{\nu}}\tilde{\chi}(N^{1/\tilde{\nu}}t), \tag{7}$$

$$U'(N) = N^{1/\tilde{\nu}}\tilde{U}'(N^{1/\tilde{\nu}}t), \tag{8}$$

where we write the reduced temperature as $t=(T-T_c)/T_c$ and $U'(N)$ is the derivative of Binder cumulant with respect to temperature. According to the finite-size scaling theory [51], the scaling functions $\tilde{m}$, $\tilde{\chi}$, and $\tilde{U}'$ are constant and $m(N)$, $\chi(N)$, and $U'(N)$ are smooth and analytic in the vicinity of the critical temperature.

C. Histogram reweighting method

The histogram reweighting method has been applied to study the critical phenomena of various spin models [43,44]. Using this, we estimate precise thermodynamic quantities depending on temperature without performing simulations at many different temperatures. If we accumulate the normalized energy histogram $h_{\beta_0}(E,m)$ from simulations directly at inverse temperature β_0, the probability $P_{\beta_0}(E,m)$ of the system having energy E and order parameter m at $h_{\beta_0}(E,m)$ could be written by

$$P_{\beta_0}(E,m) \cong h_{\beta_0}(E,m). \tag{9}$$

For the equilibrium system, the density of states, $g(E,m)$, can be expressed as the Boltzmann weight $e^{-\beta_0 E}$ using Eq. (9) as follows:

$$g(E,m) = h_{\beta_0}(E,m)e^{\beta_0 E}Z(\beta_0), \tag{10}$$

where $Z(\beta)$ is the canonical partition function. Using Eq. (10), we can easily estimate the probability $P_\beta(E,m)$ without performing simulations at the inverse temperature β as follows:

$$P_\beta(E,m) = \frac{h_{\beta_0}(E,m)e^{-\Delta\beta E}}{\sum_E h_{\beta_0}(E,m)e^{-\Delta\beta E}}, \tag{11}$$

where $\Delta\beta=\beta-\beta_0$. The expectation value of thermodynamic quantities Q_β as a function of E and m can be calculated as a continuous function β as follows:

$$\langle Q_\beta(E,m)\rangle = \sum_{E,m} Q_\beta(E,m)P_\beta(E,m). \tag{12}$$

Equations (11) and (12) are the single-histogram equations [43].

III. RESULTS

We perform Monte Carlo simulations using the histogram reweighting method for the *XY* model on growing scale-free networks, where the system size N ranges 125, 250, 500, 1000, 2000, 4000, 8000 and $q=2$ in Eq. (2). For larger systems, each run lasts 8×10^5 Monte Carlo steps (MCS). The first 2×10^5 MCS are discarded to ensure that the system has

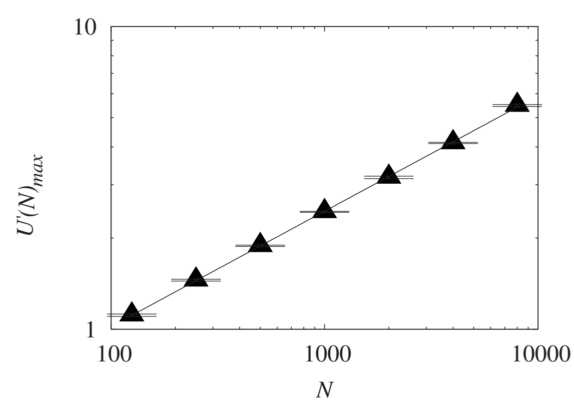

FIG. 4. Log-log plot of the maximum values of U' as a function of N for $\lambda=8.0$.

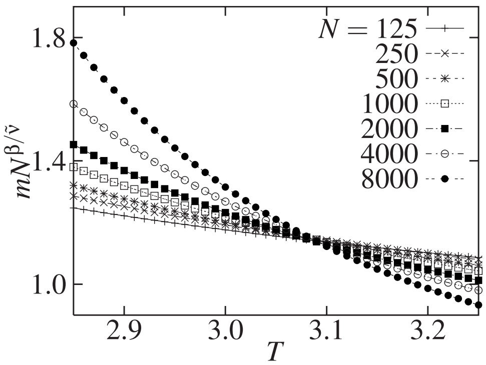

FIG. 5. Plot of the finite-size scaling of the magnetization m, using Eq. (6), yields $\beta/\tilde{\nu}$=0.442(2) with T_c=3.08(2) for λ=8.0.

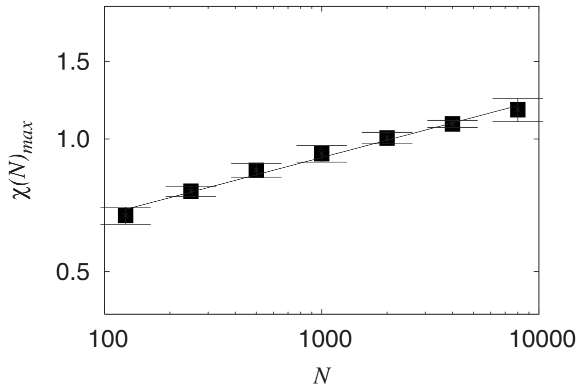

FIG. 6. Log-log plot of the maximum values of the susceptibility $\chi(N)$ as a function of N for λ=8.0.

reached a steady state, and 240 independent runs are performed and averaged by taking data at each MCS. For smaller systems, 5×10^3 MCS are discarded and the data are averaged over 1500 independent runs with 1×10^4 MCS.

The pseudocritical temperature $T_c(N)$ for the system size N is estimated by the asymptotic limit of the crossing points of the Binder cumulant $U(N)$ for successive system sizes, where $U(N)$ is calculated by the histogram reweighting method in Eqs. (11) and (12). Figure 2 shows the Binder cumulant for different system sizes as a function of temperature. For λ=8.0, the crossing point converges to $T\simeq3.0$, whereas for λ=2.1 and 3.0, it increases as a system size becomes larger.

For two system sizes N and N', the crossing point of the Binder cumulant is the pseudocritical temperature $T_c(N)$ for the system size N. As $N\to\infty$, $T_c(N)$ approaches the critical temperature. Figure 3(a) shows the double-logarithmic plot of $T_c(N)$ as a function of system size N for λ=2.1. $T_c(N)$ increases as a power law of system size N, where the power-law exponent is estimated to be 0.241(6). Figure 3(b) is a semilogarithmic plot of $T_c(N)$ for λ=3.0, indicating that $T_c(N)$ increases logarithmically as $N\to\infty$. For λ=8.0, $T_c(N)$ does not diverge in the thermodynamic limit and we can find the critical temperature by the finite-size scaling analysis of the magnetization (see Fig. 5).

At the pseudocritical temperature $T_c(N)$, the volume correlation exponent $\tilde{\nu}$ can be estimated using the Binder cumulant in Eq. (8). Figure 4 shows the double-logarithmic plot of the maximum values of U' as a function of system size N. The slope is given by $1/\tilde{\nu}$ and $\tilde{\nu}$ is estimated as 2.62(3).

We calculate the magnetization $m(N)$ at T_c using the histogram reweighting method in Eqs. (11) and (12). Then, $\beta/\tilde{\nu}$=0.442(2) and T_c=3.08(2) are estimated by the finite-size scaling of the magnetization (see Fig. 5).

Using the maximum values of the susceptibility $\chi(N)$, we are able to find $\gamma/\tilde{\nu}$. In Fig. 6, the slope of the straight line gives $\gamma/\tilde{\nu}$=0.127(4).

IV. CONCLUSIONS

To summarize, we have performed Monte Carlo simulations of the *XY* model on growing scale-free networks with various degree exponents λ by applying the histogram reweighting method. We have measured the magnetization, the susceptibility, and the Binder cumulant. The critical temperature is found to be T_c=3.08(2) for λ=8.0, while the pseudocritical temperature $T_c(N)$ diverges as $T_c(N)\sim N^{0.241(6)}$ for λ=2.1 and $T_c(N)\sim\ln N$ for λ=3.0. These divergent behaviors also have been observed in the Ising model on the scale-free networks [11–18]. For λ=8.0, we observed a second-order transition at finite temperature, and the critical exponents are found to be $\tilde{\nu}$=2.62(3), $\gamma/\tilde{\nu}$=0.127(4), and $\beta\tilde{\nu}$=0.442(2). These values satisfy the hyperscaling relation $d\nu=2\beta+\gamma$, where d is the spatial dimension and $\tilde{\nu}=d\nu$. Therefore, the *XY* model on growing scale-free networks exhibits a well-defined second-order phase transition distinct from the standard mean-field behavior.

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